POINTFREE POINTWISE SUPREMA IN ARCHIMEDEAN ℓ-GROUPS

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We generalize the concept of the pointwise supremum of real-valued functions to the pointfree setting. The concept itself admits a direct and intuitive formulation which makes no mention of points. But our aim here is to investigate pointwise suprema of subsets of $\mathcal{R}$L, the family of continuous real valued functions on a locale, or pointfree space.

Thus our setting is the category **W** of archimedean lattice-ordered groups (ℓ-groups) with designated weak order unit, with morphisms which preserve the group and lattice operations and take units to units. This is an appropriate context for this investigation because every **W**-object can be canonically represented as a subobject of $\mathcal{R}$L.

We prove the appropriate analog of the Nakano-Stone Theorem: a (completely regular) locale L has the feature that $\mathcal{R}$L is conditionally pointwise complete (σ-complete), i.e., every bounded (countable) family from $\mathcal{R}$L has a pointwise supremum in $\mathcal{R}$L, iff L is boolean (a P-locale).

It is perhaps surprising that pointwise suprema can be characterized purely algebraically, without reference to a representation. They are the context-free suprema, in the sense that the pointwise suprema are precisely those which are preserved by all morphisms out of G. We adopt the latter attribute as the final, representation-free definition of pointwise suprema.

Thus emboldened, we adopt a maximally broad definition of unconditional pointwise completeness (σ-completeness): a divisible **W**-object G is pointwise complete (σ-complete) if it contains a pointwise supremum for every subset which has a supremum in any extension. We show that the pointwise complete (σ-complete) **W**-objects are those of the form $\mathcal{R}$L for L a boolean locale (P-locale). Finally, we show that a **W**-object G is pointwise σ-complete iff it is epicomplete.

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Date: September 2014.